Random Matrix Theory in a nutshell and applications

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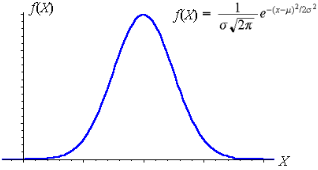
Consider a matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \dots & & a_{2N} \\ a_{31} & \dots & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ a_{M1} & a_{N2} & \dots & & & a_{MN} \end{bmatrix}$$

where the entries are random numbers:

$$A = \begin{bmatrix} * & * & * & \dots & \dots & & * \\ * & * & * & \dots & & & * \\ * & \dots & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ * & * & \dots & & & & * \end{bmatrix}$$

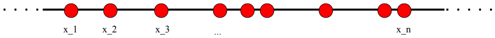
where $* =$



(Gaussian distribution, e.g.)

A few questions:

- **What about the eigenvalues?**



Their (probable) positions will depend upon the probability distribution of the entries of the matrix in a non-trivial way. There are different statistical quantities that one may study (-discrete- spectral density, gap probability, spacing, etc.)

- **What if we consider BIG matrices, possibly of ∞ dimension (appropriately rescaled)?**

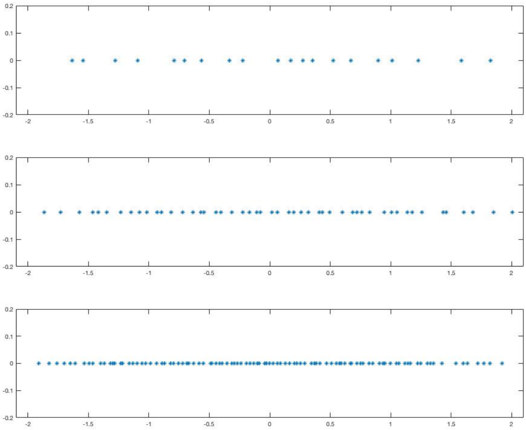


Figure: Realization of the eigenvalues of a GUE matrix of dimension $n = 20, 50, 100$.

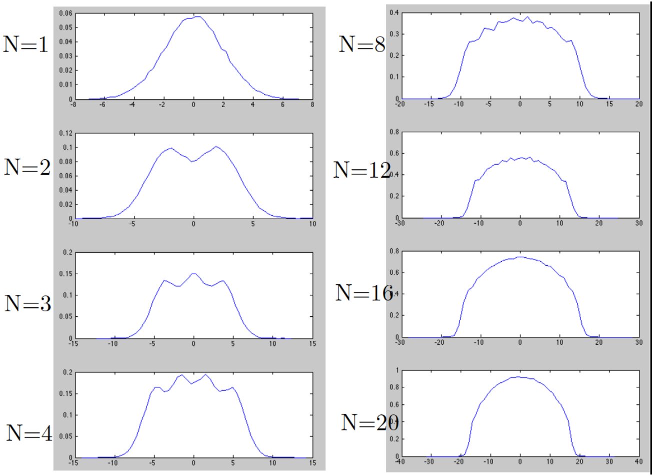
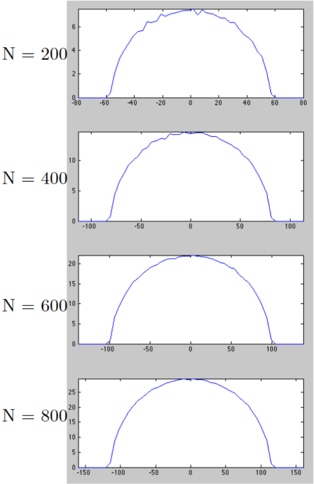


Figure: Histograms of the eigenvalues of GUE matrices as the size of the matrix increases.



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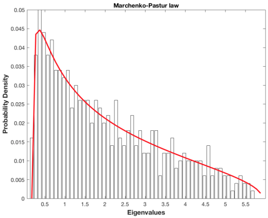
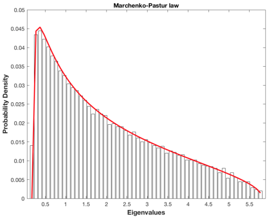
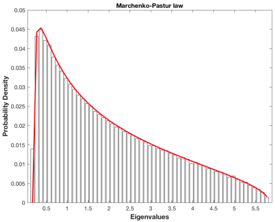


$n = 50$, $c = 2$



$n = 500$, $c = 2$

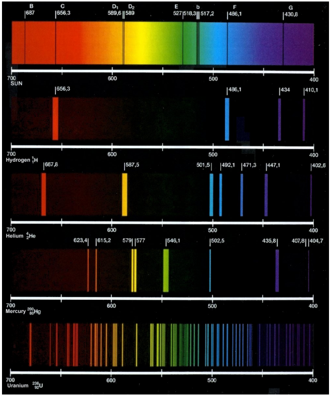


$n = 1000$, $c = 2$

Figure: Histogram of the eigenvalues of Wishart matrices as the size of the matrix increases (here $c = p/n$).

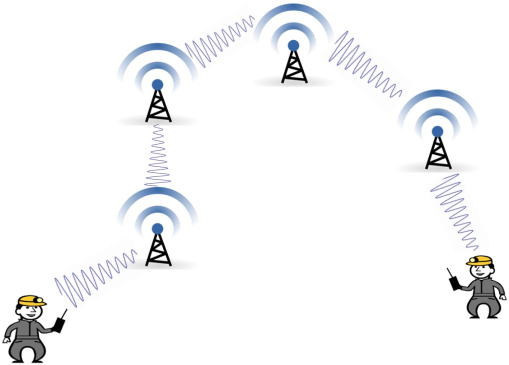
Applications

- Nuclear Physics (distribution of the energy levels of highly excited states of heavy nuclei, say uranium ${}_{92}U$)



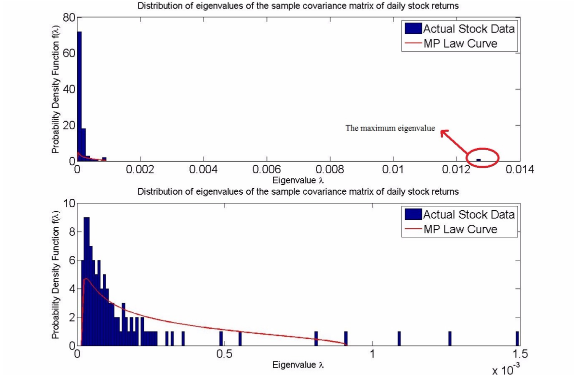
Applications

- Wireless communications



Applications

- Finance (stock markets, investment strategies)



Applications

- Very helpful in ML!

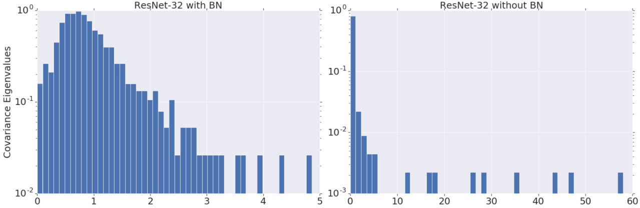


Figure: The histogram of the eigenvalues of the gradient covariance matrix $\frac{1}{n}\sum \nabla\mathcal{L}_i \nabla\mathcal{L}_i^T$ for a Resnet-32 with (left) and without (right) BN after $9k$ training steps. (from Ghorbani *et al.*, 2019)

Thanks for your attention!

"Unfortunately, no one can be told what the Matrix is.
You have to see it for yourself."

(Morpheus, "The Matrix" movie)